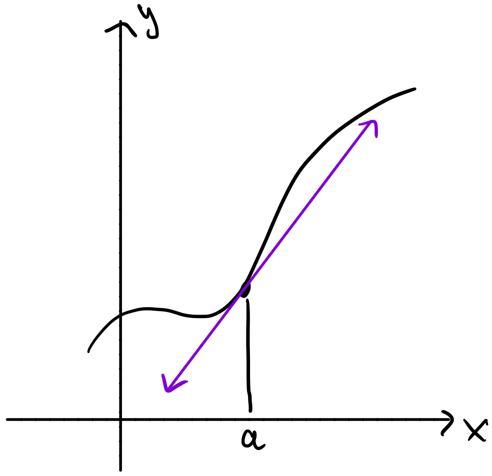
**Linear Approximation**




Basic idea:

$f(x) \approx$ tan line at $x=a$ for values of $x$ "close" to $a$

thus the tangent line is a linear approximation of $f(x)$

slope $f'(a)=m$ point $(a, f(a))$

$$y - f(a) = f'(a)(x-a)$$

$$y = f'(a)(x-a) + f(a)$$

$$\boxed{L(x) = f'(a)(x-a) + f(a)}$$

**Example 1:** Find the linearization of $f(x) = \sqrt{x+3}$ at $a = 1$ and use it to approximate the numbers $\sqrt{3.98}$ and $\sqrt{4.05}$. Are these approximations overestimates or underestimates?

$f'(x) = \frac{1}{2}(x+3)^{-1/2}$ ; $f'(1) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$ ; $f(1) = \sqrt{4} = 2$

$$L(x) = \tfrac{1}{4}(x-1)+2 = \frac{1}{4}x - \frac{1}{4} + \frac{8}{4} = \boxed{\frac{1}{4}x + \frac{7}{4}}$$

$$\sqrt{3.98} = \sqrt{0.98+3} \approx L(0.98) = \tfrac{1}{4}(0.98) + \tfrac{7}{4} = \boxed{1.995}$$

$$\sqrt{4.05} = \sqrt{1.05+3} \approx L(1.05) = \boxed{2.0125}$$

**Example 2:** Physicists commonly use linear approximations to simplify a non-linear function. Find the linear approximation of $f(x) = \sin x$. at $a=0$

$f'(x) = \cos x$ ; $f'(0) = \cos 0 = 1$

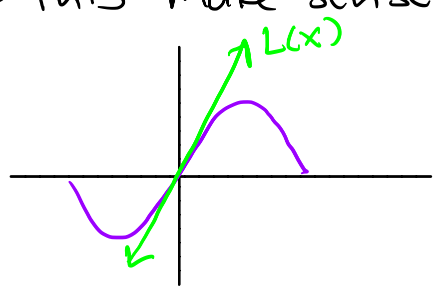
$f(0) = 0$

$L(x) = 1(x-0) + 0$

$$\boxed{L(x) = x}$$

thus $\sin x \approx x$ for $x$ "close" to zero

Does this make sense?



$$L(x) = f'(a)(x-a) + f(a)$$

**Example 3:** Use a linear approximation to estimate $\sqrt{99.8}$ in two different ways. Does your estimate differ?

(a) $f(x) = \sqrt{x}$, $a = 100$

$f'(x) = \frac{1}{2}x^{-1/2}$; $f'(100) = \frac{1}{20}$

$f(100) = 10$

$L_1(x) = \frac{1}{20}(x - 100) + 10$

$= \boxed{\frac{1}{20}x + 5}$

$\sqrt{99.8} \approx L_1(99.8)$

$= \frac{1}{20}(99.8) + 5$

$= 9.99$

not different.

(b) $f(x) = \sqrt{x + 100}$, $a = 0$

$f'(x) = \frac{1}{2}(x+100)^{-1/2}$; $f'(0) = \frac{1}{20}$

$f(0) = 10$

$L_2(x) = \frac{1}{20}(x - 0) + 10$

$= \frac{1}{20}x + 10$

$\sqrt{99.8} = \sqrt{-0.2 + 100}$

$\approx L_2(-0.2)$

$= \frac{1}{20}(-0.2) + 10$

$= 9.99$

**Example 4:** Find the linear approximation of $f(x) = e^x \cos x$ at $a = 0$. Then determine the values of $x$ for which the linear approximation is accurate to within 0.1.

$f'(x) = e^x \cos x + e^x(-\sin x)$ ; $f(0) = 1$

$f'(0) = 1 \cdot 1 + 1(-0) = 1$

$L(x) = 1(x - 0) + 1$

$= \boxed{x + 1}$

graph all three curves & see where the line falls between the upper & lower.

need $|f(x) - L(x)| < 0.1$

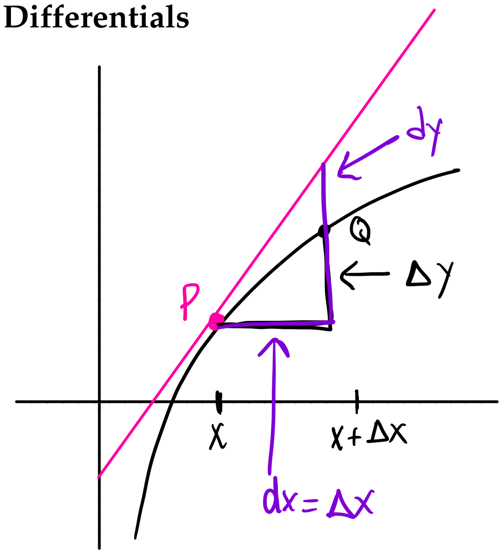
$-0.1 < e^x \cos x - (x+1) < 0.1$

$-0.1 - e^x \cos x < -(x+1) < 0.1 - e^x \cos x$

$\boxed{e^x \cos x + 0.1 > x + 1 > e^x \cos x - 0.1}$

True for $\boxed{-0.75 < x < 0.6}$

## Differentials




we want to estimate change in y or $\Delta y$.

note, the tangent line has slope $\frac{dy}{dx} = f'(x)$

$\frac{\text{rise}}{\text{run}}$

so $dy = f'(x)\, dx$

at x

note that $\Delta y \approx dy$

and as $\Delta x \to 0$, $\Delta y \to dy$

**Example 5:** Compare the values of $\Delta y$ and $dy$ if $y = f(x) = x^3 + x^2 - 2x + 1$ and $x$ changes

(a) from 2 to 2.10.

$$\Delta y = f(2.10) - f(2)$$
$$= \left((2.10)^3 + (2.10)^2 - 2(2.10) + 1\right) - \left(2^3 + 2^2 - 2(2) + 1\right)$$
$$= \boxed{1.471}$$
$$dy = f'(x)\, dx$$
$$= (3x^2 + 2x - 2)\, dx$$
$$dy = (3(2^2) + 2(2) - 2)(0.1) = \boxed{1.4}$$

(b) from 2 to 2.01.

$$\Delta y = f(2.01) - f(2)$$
$$= \left((2.01)^3 + (2.01)^2 - 2(2.01) + 1\right) - \left(2^3 + 2^2 - 2(2) + 1\right)$$
$$= \boxed{0.140701}$$
$$dy = (3(2^2) + 2(2) - 2)(0.01)$$
$$= \boxed{0.14}$$

(c) What happens to $\Delta y$ and $dy$ as $\Delta x$ decreases?

$$\Delta y \to dy$$

**Example 6:** Find the differential of the function.

(a) $y = \cos \pi x$

$dy = -\sin(\pi x) \cdot \frac{d}{dx} \pi x$

$\boxed{dy = -\pi \sin(\pi x)\, dx}$

(b) $y = \frac{1}{(1+2r)^4} = (1+2r)^{-4}$

$dy = -4(1+2r)^{-5} \cdot \frac{d}{dr}(1+2r)$

$dy = -4(1+2r)^{-5}(2)\, dr$

$\boxed{dy = \frac{-8}{(1+2r)^5}\, dr}$

**Example 7:** Find the differential $dy$ and evaluate $dy$ for the given values of $x$ and $dx$.

(a) $y = x^3 - 6x^2 + 5x - 7$ , $x = -2$, $dx = 0.1$

$dy = (3x^2 - 12x + 5)\, dx$

$dy = (3(-2)^2 - 12(-2) + 5)(0.1)$

$\boxed{dy = 4.1}$

(b) $y = 1/(x+1)$, $x = 1$, $dx = -0.01$

$y = (x+1)^{-1}$

$dy = -1(x+1)^{-2}\, dx$

$dy = \frac{-1}{(x+1)^2}\, dx$

$dy = \frac{-1}{(1+1)^2}(-0.01)$

$\boxed{dy = 0.0025}$

**Example 8:** The radius of a sphere was measured and found to be 21 cm with a possible error of at most 0.05 cm. What is the maximum error in using this value of the radius to compute the volume of the sphere?

$\underline{\underline{dr}}$

$V = \frac{4}{3}\pi r^3$

$dV = 4\pi r^2 \, dr$

$dV = 4\pi(21)^2(0.05)$

$\boxed{dV = 88.2\pi \text{ cm}^3}$

$\boxed{dV \approx 277.088 \text{ cm}^3}$

**Example 9:** The radius of a circular disk is given at 24 cm with a maximum error in measurement of at most 0.2 cm.

$dr = 0.2$

(a) Use differentials to estimate the maximum error in the calculated area of the disc. Does this error seem large?

$A = \pi r^2$

$dA = 2\pi r \, dr$

$dA = 2\pi(24)(0.2)$

$\boxed{dA = 9.6\pi \text{ cm}^2}$

$\boxed{dA \approx 30.159 \text{ cm}^2}$

I think it seems large

(b) What is the relative error? What is the percent error? Does the error (still) seem large?

$$\frac{\Delta A}{A} \approx \frac{dA}{A} = \frac{2\pi r \, dr}{\pi r^2} = \frac{2\pi(24)0.2}{\pi(24)^2} = \frac{0.4}{24} = \boxed{0.01\overline{6}} \text{ OR } \boxed{1.\overline{6}\%}$$

this doest sound so bad.